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NOTE

On the Number of Planes in Neumaier's A_8-Geometry

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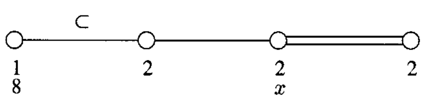
In one of his papers [2], A. Neumaier constructed a rank 4 incidence geometry on which the alternating group of degree 8 acts flag-transitively. This geometry is quite important since its point residue is the famous A_7-geometry which is known to be the only flag-transitive locally classical C_3-geometry which is not a polar space (see [1]). By counting chambers, we prove that the A_8-geometry has 70 planes. This can be found in a paper of Pasini's [4] without proof, but Neumaier's original paper only mentions 35 planes. © 2001 Academic Press

Key Words: Neumaier's geometry; diagram geometry; alternating group.

1. SETTING

This note is concerned with the rank 4 geometry Γ having the diagram



The nodes in this diagram represent the 4 types of elements of Γ, which we call, from left to right, *points*, *lines*, *planes*, and *hyperplanes*. A construction of this geometry can be found in [3]. We shall only use the information given in the above diagram. This includes the orders of every type of element and the number of points of Γ.

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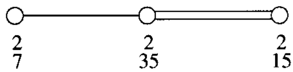
2. THE A_8-GEOMETRY HAS 70 PLANES

If we denote the number of planes in Γ by x, we have that the number of chambers in Γ equals x times the number of chambers in the residue Γ' of a plane. The number of chambers in Γ is also 8 times the number of chambers in the point residue Γ'' of Γ. By [1], we know that this residue must be the A_7-geometry with the diagram



Hence the number of chambers in Γ'' is 7 times the number of chambers in the point-line residue of Γ. The diagram of Γ shows that this residue is a generalized quadrangle of order 2. It is well known (see [5]) that only one such generalized quadrangle exists, namely the symplectic quadrangle $W(2)$. The number of chambers in $W(2)$ is 45, implying that the number of chambers in Γ is $8 \cdot 7 \cdot 45 = 2520$.

The plane residue Γ' must have the diagram

The diagram, together with the orders, tells us that the number of points in such a residue is 4 and the number of lines is 6. To find the number of hyperplanes in this residue, we remark that its diagram implies that this number is the same as the number of hyperplanes incident with a plane in a point-line residue. As above, the point-line residue is isomorphic to the symplectic quadrangle $W(2)$, which has 3 lines incident with every point. Hence the number of chambers in the residue Γ' is $3 \cdot 6 \cdot 2 = 36$.

Finally we get $x \cdot 36 = 2520$, implying $x = 70$.

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